Math 113 (Calculus II)
Midterm Exam 1
Solutions

Instructions:

- Work on scratch paper will not be graded.
- For questions 6 to 11, show **all** your work in the space provided. Full credit will be given only if the necessary work is shown justifying your answer. Please write neatly.
- Should you have need for more space than is allotted to answer a question, use the back of the page the problem is on and indicate this fact.
- Simplify your answers. Expressions such as ln(1), e^0, $\sin(\pi/2)$, etc. must be simplified for full credit.
- Calculators are not allowed.

For Instructor use only.

#	Possible	Earned
MC	15	
6a	15	
6d	10	
7	5	
8	10	
9a	5	
9b	5	
Sub	65	

#	Possible	Earned
9c	5	
9d	5	
9e	5	
9f	5	
10	5	
11a	5	
11b	5	
Sub	35	
Total	100	

Multiple Choice. Fill in the answer to each problem on your computer-scored answer sheet. Make sure your name, section and instructor are on that sheet.

1. Find the volume of the solid obtained by rotating the region bounded by the curves $y = \sec(x)$, $y = 0$, $x = 0$, $x = \dfrac{\pi}{4}$ about the x-axis.

   a) 1   b) 2

   c) 3   d) π

   e) $\dfrac{\pi}{2}$ f) $\dfrac{\pi}{3}$

   g) None of the above

   ANSWER: D

2. Find the average value of the function $f(x) = \sqrt[3]{x}$ on the interval $[1,\ 8]$.

   a) 12   b) $\dfrac{12}{7}$

   c) $\dfrac{45}{4}$ d) $\dfrac{3}{2}$

   e) $\dfrac{3}{14}$ f) $\dfrac{45}{28}$

   g) None of the above

   ANSWER: F

3. If $f(1) = 2, f(4) = 7, f'(1) = 5, f'(4) = 3$, and $f''(x)$ is continuous, what is $\displaystyle\int_1^4 x f''(x)\, dx$?

   a) 9   b) -2

   c) 12   d) -1

   e) 2   f) None of the above.

   ANSWER: E

4. What is $\displaystyle\int_0^{\frac{\pi}{4}} \sin^2(2\theta)\, d\theta$?

   a) 1   b) 0

   c) $\dfrac{1}{2}$ d) $\dfrac{\pi}{8}$

   e) $\dfrac{\pi - 2}{8}$ f) None of the above

   ANSWER: D

5. What is the best form for the partial fraction decomposition of $\frac{2x+1}{(x+1)^3(x^2+4)^2}$?

a) $\frac{A}{(x+1)^3} + \frac{Bx+C}{(x^2+4)^2}$

b) $\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{Dx+E}{x^2+4} + \frac{Fx+G}{(x^2+4)^2}$

c) $\frac{A}{x+1} + \frac{B}{(x+1)^3} + \frac{Cx+D}{x^2+4}$

d) $\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{D}{x^2+4} + \frac{E}{(x^2+4)^2}$

e) $\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{Dx+E}{x^2+4} + \frac{Fx+G}{(x+1)^3(x^2+4)^2}$

f) None of the above

ANSWER: B

Free response: Write your solution and answer in the space provided. Answers not placed in this space will be ignored.

6. Consider the region between the curves $y = 5x$ and $y = x^2$ in the first quadrant.

(a) (15 points) Set up an integral for the area of the region bounded by the curves. DO NOT EVALUATE.

ANSWER: $\int_0^5 5x - x^2\,dx$ or $\int_0^{25} \sqrt{y} - \frac{y}{5}\,dy$

(b) Set up an integral for the volume obtained when the region is rotated about the x-axis. DO NOT EVALUATE.

ANSWER: $\int_0^5 \pi((5x)^2 - x^4)\,dx$ or $\int_0^{25} 2\pi y(\sqrt{y} - \frac{y}{5})\,dy$

(c) Set up an integral for the volume obtained when the region is rotated about the y-axis. DO NOT EVALUATE.

ANSWER: $\int_0^{25} \pi(y - \frac{y^2}{25})\,dy$ or $\int_0^5 2\pi x(5x - x^2)\,dx$

(d) (10 points) Set up an integral for the volume obtained when the region is rotated about the line $y = -2$. DO NOT EVALUATE.

ANSWER: $\int_0^5 \pi[(5x+2)^2 - (x^2+2)^2]\,dx$ or $\int_0^{25} 2\pi(y+2)(\sqrt{y} - \frac{y}{5})\,dy$

(e) Set up an integral for the volume obtained when the region is rotated about the line $x = -3$. DO NOT EVALUATE.

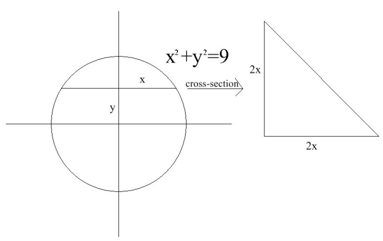
ANSWER: $\int_0^{25} \pi[(\sqrt{y}+3)^2 - (\frac{y}{5}+3)^2]\,dy$ or $\int_0^5 2\pi(x+3)(5x - x^2)\,dx$

7. (5 points) A 12-ft chain weighs 36 lbs and hangs over the edge of a 20 ft high building. How much work is done in pulling the chain to the top of the building?

ANSWER:

The chain weighs $\frac{36}{12}$ lbs per foot, or 3 lbs per foot. The work to raise one slice of length dz is force times distance, so it is $W_{slice} = 3z\,dz$. Thus the total work is $W = \int_0^{12} 3z\,dz = \frac{3}{2}z^2|_0^{12} = \frac{3}{2}(12)^2 = 3(72) = 216$ ft-lb.

8. (10 points) The base of a solid is a circular disk with radius 3. Find the volume of the solid if parallel cross-sections perpendicular to the base are isosceles right triangles with one of the two equal sides lying along the base.



Answer:

The volume of a slice is the area of the triangle times the width of the triangle, so it is $V_{slice} = \frac{1}{2}(2x)^2\,dy = 2x^2\,dy$. But we need x in terms of y, so we use the equation of the circle to get $x = \sqrt{9 - y^2}$. Thus our volume is:

$$\begin{aligned} V &= 2\int_0^3 2(\sqrt{9-y^2})^2\,dy \\ &= 4\int_0^3 9 - y^2\,dy \\ &= 4(9y - \frac{1}{3}y^3)_0^3 \\ &= 4(9(3) - \frac{1}{3}27) \\ &= 4(27-9) \\ &= 72 \end{aligned}$$

9. Integrate the following and show all of your work:

(a) (5 points) $\int \sin^6 x \cos^3 x \, dx$

Answer:

$$\begin{aligned}
\int \sin^6 x \cos^3 x \, dx &= \int \sin^6 x (1 - \sin^2 x) \cos x \, dx \text{ let } u = \sin x, \text{ then } du = \cos x \, dx \\
&= \int u^6 (1 - u^2) \, du \\
&= \int u^6 - u^8 \, du \\
&= \frac{1}{7} u^7 - \frac{1}{9} u^9 + C \\
&= \frac{1}{7} \sin^7 x - \frac{1}{9} \sin^9 x + C
\end{aligned}$$

(b) (5 points) $\int t^5 \ln t \, dt$

Answer:

Integration by parts: $u = \ln t \quad dv = t^5 \, dt$; $du = \frac{1}{t} \, dt \quad v = \frac{1}{6} t^6$

$$\begin{aligned}
\int_2^6 t^5 \ln t \, dt &= \frac{1}{6} t^6 \ln t - \int \frac{1}{6} t^5 \, dt \\
&= \left[\frac{1}{6} t^6 \ln t - \frac{1}{36} t^6 \right]_2^6 \\
&= \frac{1}{6} (6)^6 \ln 6 - \frac{1}{36} (6)^6 - \frac{1}{6} (2)^6 \ln 2 + \frac{1}{36} (2)^6
\end{aligned}$$

(c) (5 points) $\int \frac{\ln(\ln x)}{x \ln x}\,dx$

Answer:

Let $u = \ln x$. Then $du = \frac{1}{x}\,dx$. Thus we have

$$\begin{aligned}
\int \frac{\ln\ln x}{x\ln x}\,dx &= \int \frac{\ln u}{u}\,du \text{ Let } v = \ln u, \text{ then } dv = \frac{1}{u}\,du \\
&= \int v\,dv \\
&= \frac{1}{2}v^2 + C \\
&= \frac{1}{2}(\ln u)^2 + C \\
&= \frac{1}{2}(\ln\ln x)^2 + C
\end{aligned}$$

(d) (5 points) $\int x \sin 7x\,dx$

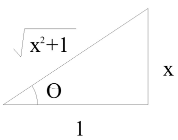
Answer:

Integration by parts: $\begin{array}{ll} u = x & dv = \sin 7x\,dx \\ du = dx & v = \frac{-1}{7}\cos 7x \end{array}$

$$\begin{aligned}
\int x\sin 7x\,dx &= \frac{-x}{7}\cos 7x - \int \frac{-1}{7}\cos 7x\,dx \\
&= \frac{-x}{7}\cos 7x + \frac{1}{49}\sin 7x + C
\end{aligned}$$

(e) (5 points) $\int x^3\sqrt{x^2+1}\,dx$

Answer: Let $x = \tan\theta$, then $dx = \sec^2\theta\,d\theta$.



$$
\begin{aligned}
\int x^3\sqrt{x^2+1}\,dx &= \int \tan^3\theta(\tan^2\theta+1)^{1/2}\sec^2\theta\,d\theta \\
&= \int \tan^3\theta\sec^3\theta\,d\theta \\
&= \int (\tan^2\theta\sec^2\theta)\tan\theta\sec\theta\,d\theta \\
&= \int (\sec^2\theta+1)\sec^2\theta\tan\theta\sec\theta\,d\theta \text{ Let } u=\sec\theta, du=\sec\theta\tan\theta\,d\theta \\
&= \int (u^2+1)u^2\,du \\
&= \int u^4+u^2\,du \\
&= \frac{1}{5}u^5+\frac{1}{3}u^3+C \\
&= \frac{1}{5}\sec^5\theta+\frac{1}{3}\sec^3\theta+C \\
&= \frac{1}{5}(x^2+1)^{5/2}+\frac{1}{3}(x^2+1)^{3/2}+C
\end{aligned}
$$

(f) (5 points) $\int e^{2\theta}\cos 4\theta\,d\theta$

Answer:

Integration by parts: $\begin{array}{ll} u = e^{2\theta} & dv = \cos 4\theta\,d\theta \\ du = 2e^{2\theta}\,d\theta & v = \frac{1}{4}\sin 4\theta \end{array}$

$$\int e^{2\theta}\cos 4\theta\,d\theta = \frac{1}{4}e^{2\theta}\sin 4\theta - \frac{1}{2}\int e^{2\theta}\sin 4\theta\,d\theta$$

Integration by parts again: $\begin{array}{ll} u = e^{2\theta} & dv = \sin 4\theta\,d\theta \\ du = 2e^{2\theta}\,d\theta & v = \frac{-1}{4}\cos 4\theta \end{array}$

$$
\begin{aligned}
\int e^{2\theta}\cos 4\theta\,d\theta &= \frac{1}{4}e^{2\theta}\sin 4\theta - \frac{1}{2}\left[\frac{-1}{4}e^{2\theta}\cos 4\theta + \int \frac{1}{2}e^{2\theta}\cos 4\theta\,d\theta\right] \\
&= \frac{1}{4}e^{2\theta}\sin 4\theta + \frac{1}{8}e^{2\theta}\cos 4\theta - \frac{1}{4}\int e^{2\theta}\cos 4\theta\,d\theta \\
\frac{5}{4}\int e^{2\theta}\cos 4\theta\,d\theta &= \frac{1}{4}e^{2\theta}\sin 4\theta + \frac{1}{8}e^{2\theta}\cos 4\theta + C \\
\int e^{2\theta}\cos 4\theta\,d\theta &= \frac{4}{5}\left[\frac{1}{4}e^{2\theta}\sin 4\theta + \frac{1}{8}e^{2\theta}\cos 4\theta\right] + C \\
&= \frac{1}{5}e^{2\theta}\sin 4\theta + \frac{1}{10}e^{2\theta}\cos 4\theta + C
\end{aligned}
$$

10. (5 points) A force of 12 lb is required to hold a spring stretched 3 in. beyond its natural length. How much work is done in stretching it from its natural length to 4 in. beyond its natural length?

ANSWER: Recall that we need our units in feet, so 3 in $= \frac{1}{4}$ foot.

$$\begin{aligned} F &= kx \\ 12 &= \frac{k}{4} \\ k &= 48 \end{aligned}$$

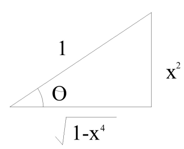
Thus we have that the work to stretch the spring to 4 in $= \frac{1}{3}$ feet is:

$$\begin{aligned} W &= \int_0^{\frac{1}{3}} 48x\,dx \\ &= 24x^2|_0^{\frac{1}{3}} \\ &= 24(\frac{1}{9}) \\ &= \frac{8}{3}\text{ft-lb} \end{aligned}$$

11. Integrate the following:

(a) (5 points) $\int x\sqrt{1-x^4}\,dx$

Answer:



Let $x^2 = \sin\theta$. Then $2x\,dx = \cos\theta\,d\theta$.

$$\begin{aligned} \int x\sqrt{1-x^4}\,dx &= \int \frac{1}{2}\sqrt{1-\sin^2\theta}\cos\theta\,d\theta \\ &= \frac{1}{2}\int \cos^2\theta\,d\theta \\ &= \frac{1}{2}\int \frac{1}{2}(1+\cos 2\theta)\,d\theta \\ &= \frac{1}{4}\int 1+\cos 2\theta\,d\theta \\ &= \frac{1}{4}(\theta + \frac{1}{2}\sin 2\theta) + C \\ &= \frac{1}{4}[\theta + \frac{1}{2}(2\sin\theta\cos\theta)] + C \\ &= \frac{1}{4}(\sin^{-1}x^2 + x^2\sqrt{1-x^4}) + C \end{aligned}$$

(b) (5 points) $\displaystyle\int_2^3 \frac{2x+3}{(x-1)(x+4)}\,dx$

ANSWER:

First we need to find the partial fraction decomposition of the integrand.

$$\begin{aligned}\frac{2x+3}{(x-1)(x+4)} &= \frac{A}{x-1}+\frac{B}{x+4}\\ 2x+3 &= A(x+4)+B(x-1)\end{aligned}$$

At $x=-4$ we have $2(-4)+3=-5B$. Thus $B=1$. At $x=1$ we have $2+3=5A$, so $A=1$ also. Thus our integral becomes:

$$\begin{aligned}\int_2^3 \frac{2x+3}{(x-1)(x+4)}\,dx &= \int_2^3 \frac{1}{x-1}+\frac{1}{x+4}\,dx\\ &= \ln|x-1|+\ln|x+4|\big|_2^3\\ &= \ln 2+\ln 7-(\ln 1+\ln 6)\\ &= \ln 2+\ln 7-\ln 6\\ &= \ln\frac{14}{6}\\ &= \ln\frac{7}{3}\end{aligned}$$

END OF EXAM